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Equivalent Representations of Nonuniform Transmission Lines Based on the Extended Kuroda's Identity

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Abstract—**Kuroda's identity may be extended to circuits consisting of lumped reactance elements and nonuniform transmission lines. It is shown that these circuits are equivalent to circuits consisting of cascade connections of nonuniform transmission lines whose characteristic impedance distributions are different from original ones, lumped reactance elements, and ideal transformers. If a characteristic impedance distribution $W(x)$ of an original nonuniform transmission line is given, a characteristic impedance distribution $Z(x)$ of a transformed nonuniform transmission line may be uniquely obtained using $W(x)$. Moreover, by using these equivalent transformations, network functions of these transformed nonuniform transmission lines can be derived exactly.**

I. Introduction

It is well known that nonuniform transmission lines show superior transmission responses than the ones of uniform transmission lines. But, it is quite difficult to find the exact network functions of general nonuniform transmission lines from the telegrapher's equation except some nonuniform transmission lines [1]–[12].

On the other hand, we showed that the network functions of a class of nonuniform transmission lines can be exactly derived by using extended Kuroda's identities to mixed lumped and distributed circuits [13].

In this paper, we show a method to extend Kuroda's identities for mixed lumped and nonuniform distributed circuits. Nonuniform transmission lines are shown in the limit of cascaded transmission lines (CTL's) when line length of unit element (UE) approaches zero. By applying Kuroda's identities to circuits consisting of a single stub and CTL's n times, we can show that Kuroda's identities can be extended to circuits consisting of a lumped reactance element and a nonuniform transmission line as the limit case. The transformed circuit becomes the one consisting of a cascade connection of a nonuniform transmission line, a lumped reactance element, and an ideal transformer. Namely, if a characteristic impedance distribution $W(x)$ of an original nonuniform transmission line can be integrated, a characteristic impedance distribution $Z(x)$ of a transformed nonuniform transmission line may be uniquely obtained using $W(x)$. Also, if an exact network function of an original nonuniform transmission line is known, a network function of a transformed nonuniform transmission line can be obtained exactly. We derive exact network functions of several nonuniform transmission lines by applying extended Kuroda's identity to nth order binomial form nonuniform transmission line, exponential transmission line, and hyperbolic secant squared tapered transmission line.

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II. Representation of Nonuniform Transmission Lines

Cascaded transmission lines (CTL's) are shown in Fig. 1(a), where line length and a characteristic impedance of a lossless uniform transmission line (UE) are l/n and W_i $(i=1,2,\cdots,n)$, respectively.

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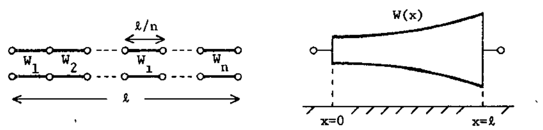


(a) (b)

Fig. 1. Cascaded transmission lines and a nonuniform transmission line.

TABLE I
CHARACTERISTIC IMPEDANCE DISTRIBUTION OF NONUNIFORM TRANSMISSION LINE

	Coefficients a_m	Characteristic Impedance Distribution W(x)	Note
1	$a_m = 0$ (m=1,2,---)	$W_u(x) = W_0$	uniform transmission line
2	$a_1 = 1/h$ $a_m = 0$ (m=2,3,---)	$W_\ell(x) = W_0(1 + \frac{1}{h}\frac{x}{\ell})$	linearly tapered transmission line h : taper coefficient
3	$a_1 = 2/h$, $a_2 = 1/h^2$ $a_m = 0$ (m=3,4,---)	$W_p(x) = W_0(1 + \frac{1}{h}\frac{x}{\ell})^2$	parabolic tapered transmission line
4	$a_m = \binom{n}{m}(\frac{1}{h})^m$ (m=1,2,---,n) $a_{n+r} = 0$ (r=1,2,---)	$W_b(x) = W_0(1 + \frac{1}{h}\frac{x}{\ell})^n$	n-th order binomial form nonuniform transmission line
5	$a_m = \frac{(\delta\ell)^m}{m!}$ (m=1,2,---)	$W_e(x) = W_0 \exp(\delta x)$	exponential transmission line δ : taper coefficient
6	$a_{2m} = \frac{(\delta\ell)^{2m}}{m!}$ $a_{2m-1} = 0$ (m=1,2,---)	$W_H(x) = W_0 \exp[(\delta x)^2]$	Hermite line
7	$a_{2m} = \frac{(2\delta\ell)^{2m}}{2\,(2m)!}$ $a_{2m-1} = 0$ (m=1,2,---)	$W_{ch}(x) = W_0 \cosh^2(\delta x)$	hyperbolic cosine squared tapered transmission line
8	$a_m = (-1)^{m-1}\frac{(2m-3)!!}{m!\,2^m}(\frac{1}{h})^m$ (m=1,2,---)	$W_r(x) = W_0\sqrt{1 + \frac{1}{h}\frac{x}{\ell}}$	square root tapered transmission line

Here, we define the characteristic impedance of the ith UE of the circuit shown in Fig. 1(a) as follows:

$$W_i = W_0\left[1 + \frac{a_1}{n}(i-1) + \frac{a_2}{n^2}(i-1)^2 + \cdots + \frac{a_m}{n^m}(i-1)^m + \cdots\right] \quad (1)$$

where W_0 is the characteristic impedance of the first UE and a_m $(m=1,2,\cdots)$ are constants. Also, the coordinates x of the ith UE is given as follows [13]:

$$x = \frac{i}{n} l. \quad (2)$$

By setting coefficients a_m appropriate values and proceeding to the limit $n \to \infty$ [13], we can obtain the various characteristic impedance distribution $W(x)$ of nonuniform transmission lines from (1)

$$W(x) = \lim_{n\to\infty} W_i = W_0\left[1 + a_1\left(\frac{x}{l}\right) + a_2\left(\frac{x}{l}\right)^2 + \cdots + a_m\left(\frac{x}{l}\right)^m + \cdots\right]. \quad (3)$$

Several examples are given in Table I.

III. EXTENDED KURODA'S IDENTITIES FOR MIXED LUMPED AND NONUNIFORM DISTRIBUTED CIRCUITS

By using Kuroda's identity n times to the circuit consisting of a single short-circuited stub and CTL's, we can obtain the equivalent transformation I in Table II [13]. Where k_j is the transformer ratio obtained after the jth Kuroda's identity, and given by

$$k_j = 1 + \frac{1}{L}\sum_{i=1}^{j} W_i \qquad (j=1,2,\cdots,n). \quad (4)$$

Similarly, for the circuit consisting of a single open-circuited stub and CTL's, we can obtain the equivalent transformation II in Table III [13]. Kuroda's identities in Tables II and III are dual transformations.

By substituting (1) in (4), we get

$$k_j = 1 + \frac{W_0}{L}\left[j + \frac{a_1}{n}\left\{\frac{(j-1)^2}{2} + \frac{(j-1)}{2}\right\} + \frac{a_2}{n^2}\left\{\frac{(j-1)^3}{3} + \frac{(j-1)^2}{2} + \frac{(j-1)}{6}\right\} + \cdots + \frac{a_m}{n^m}\left\{\frac{(j-1)^{m+1}}{m+1} + \frac{(j-1)^m}{2} + \cdots\right\} + \cdots\right]. \quad (5)$$

By using the same techniques shown in [13], we obtain

$$k(x) = \lim_{n\to\infty} k_j = 1 + \frac{W_0}{L_0}\left[\left(\frac{x}{l}\right) + \frac{a_1}{2}\left(\frac{x}{l}\right)^2 + \frac{a_2}{3}\left(\frac{x}{l}\right)^3 + \frac{a_m}{m+1}\left(\frac{x}{l}\right)^{m+1} + \cdots\right] = 1 + \frac{1}{L_0}\int_0^{x/l} W(\lambda)\, d\left(\frac{\lambda}{l}\right) \quad (6)$$

and

$$Z(x) = \lim_{n\to\infty} Z_j = \lim_{n\to\infty}\frac{W_j}{k_{j-1}k_j} = \frac{W(x)}{k(x)^2} = \frac{W(x)}{\left[1 + \frac{1}{L_0}\int_0^{x/l} W(\lambda)\, d\left(\frac{\lambda}{l}\right)\right]^2} \quad (7)$$

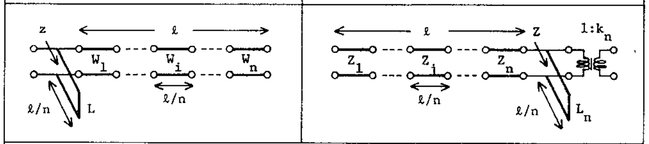
where

$$L = nL_0. \quad (8)$$

Namely, a characteristic impedance distribution $Z(x)$ of a transformed nonuniform transmission line may be uniquely expressed with an integration of a characteristic impedance distribution $W(x)$ of an original nonuniform transmission line. The impedance z and Z of the single short-circuited

TABLE II
KURODA'S IDENTITY I

Original circuit	Equivalent circuit



Formula

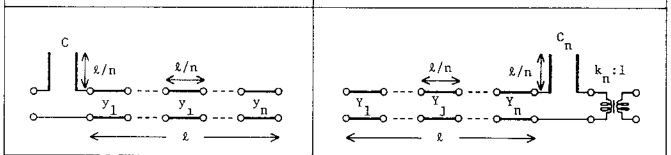
$k_j = 1 + \frac{1}{L}\sum_{i=1}^{j} W_i$ (j=1,2,---,n) , $k_0 = 1$

$Z_j = \frac{W_j}{k_{j-1}\, k_j}$ (j=1,2,---,n) , $L_n = \frac{L}{k_n}$

L , L_n , Z_j : characteristic impedances

TABLE III
KURODA'S IDENTITY II

Original circuit	Equivalent circuit



Formula

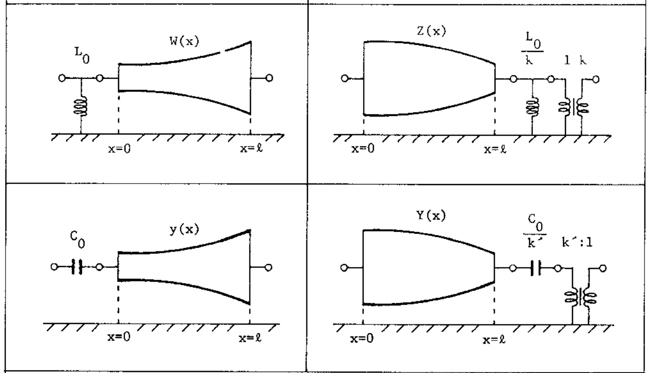
$k_j = 1 + \frac{1}{C}\sum_{i=1}^{j} y_i$ (j=1,2,---,n) , $k_0 = 1$

$Y_j = \frac{y_j}{k_{j-1}\, k_j}$ (j=1,2,---,n) , $C_n = \frac{C}{k_n}$

C , C_n , y_i , Y_j : characteristic admittances

TABLE IV
EXTENDED KURODA'S IDENTITIES

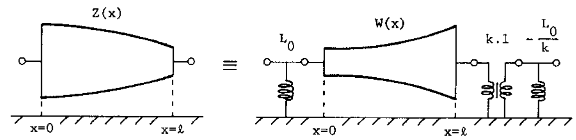
Original circuit	Equivalent circuit



Formula

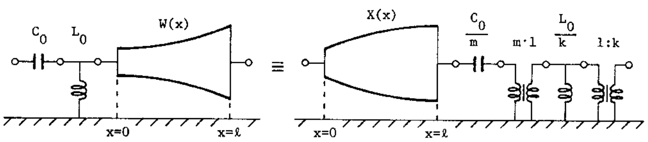
$k(x) = 1 + \frac{1}{L_0}\int_0^{\frac{x}{\ell}} W(\lambda)\, d\left(\frac{\lambda}{\ell}\right)$, $Z(x) = \frac{W(x)}{k(x)^2}$, $k = k(x)|_{x=\ell}$

$k'(x) = 1 + \frac{1}{C_0}\int_0^{\frac{x}{\ell}} y(\lambda)\, d\left(\frac{\lambda}{\ell}\right)$, $Y(x) = \frac{y(x)}{k'(x)^2}$, $k' = k'(x)|_{x=\ell}$



(a) (b)

Fig. 2. The new nonuniform transmission line and its equivalent circuit.



(a) (b)

Fig. 3. Extended Kuroda's identity for a circuit consisting of lumped reactance elements and a nonuniform transmission line.

stubs in Table II yield

$$\lim_{n\to\infty} z = \lim_{n\to\infty}\left[jnL_0 \tan\frac{\beta l}{n}\right] = jL_0\beta l \quad (9)$$

$$\lim_{n\to\infty} Z = \lim_{n\to\infty}\left[j\frac{nL_0}{k_n} \tan\frac{\beta l}{n}\right] = j\frac{L_0}{k}\beta l \quad (10)$$

where

$$k = k(x)|_{x=l} \quad (11)$$

and β is the phase constant. The single short-circuited stubs become lumped inductances. Similarly, the single open-circuited stubs in Table III become lumped capacitors. Extended Kuroda's identities for mixed lumped and nonuniform distributed circuits are shown in Table IV. Where

$$k' = 1 + \frac{1}{C_0}\int_0^l y(x)\, d\left(\frac{x}{l}\right) \quad (12)$$

and $y(x)$ and $Y(x)$ are characteristic admittance distributions of nonuniform transmission lines.

From circuits in Table IV, the equivalent circuit of the transformed nonuniform transmission line with $Z(x)$ can be expressed as the mixed lumped and nonuniform distributed circuit shown in Fig. 2(b). Therefore, if an exact network function of an original nonuniform transmission line with $W(x)$ is given, a network function of a transformed nonuniform transmission line with $Z(x)$ can be exactly derived as follows:

$$\begin{pmatrix} \mathbb{A}_Z & \mathbb{B}_Z \\ \mathbb{C}_Z & \mathbb{D}_Z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{jL_0\beta l} & 1 \end{pmatrix} \begin{pmatrix} \mathbb{A}_W & \mathbb{B}_W \\ \mathbb{C}_W & \mathbb{D}_W \end{pmatrix} \cdot \begin{pmatrix} k & 0 \\ 0 & \frac{1}{k} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{k}{jL_0\beta l} & 1 \end{pmatrix}. \quad (13)$$

Where, $\mathbb{A}_Z - \mathbb{D}_Z$ and $\mathbb{A}_W - \mathbb{D}_W$ are elements of chain matrices of nonuniform tranmission line of $Z(x)$ and $W(x)$, respectively.

Furthermore, by using equivalent transformations shown in Table IV, equivalent relations shown in Fig. 3 (a) and (b) are applicable. Where the characteristic impedance distribution $X(x)$ of the transformed nonuniform transmission line is given by

$$X(x)\equiv\frac{1}{Y(x)}\equiv\frac{m(x)^2}{y(x)}=\frac{k'(x)^2}{y(x)}$$

$$=\frac{\left[1+\frac{1}{C_0}\int_0^{x/l}y(\lambda)d\left(\frac{\lambda}{l}\right)\right]^2}{y(x)}$$

$$=\frac{\left[1+\frac{1}{C_0}\int_0^{x/l}\frac{1}{Z(\lambda)}d\left(\frac{\lambda}{l}\right)\right]^2}{\frac{1}{Z(x)}}$$

$$=\frac{\left\|1+\frac{1}{C_0}\int_0^{x/l}\left\{\frac{\left[1+\frac{1}{L_0}\int_0^{\lambda/l}W(x)d\left(\frac{x}{l}\right)\right]^2}{W(\lambda)}\right\}d\left(\frac{\lambda}{l}\right)\right\|^2}{\frac{\left[1+\frac{1}{L_0}\int_0^{x/l}W(\lambda)d\left(\frac{\lambda}{l}\right)\right]^2}{W(x)}} \tag{14}$$

and

$$m=1+\frac{1}{C_0}\int_0^1\left\{\frac{\left[1+\frac{1}{L_0}\int_0^{x/l}W(\lambda)d\left(\frac{\lambda}{l}\right)\right]^2}{W(x)}\right\}d\left(\frac{x}{l}\right). \tag{15}$$

We can repeat such equivalent procedures again and again, so we can find exact network functions of a number of nonuniform transmission lines.

Here, we apply extended Kuroda's identity to several circuits consisting of lumped inductances and nonuniform transmission lines, and get exact network functions of transformed nonuniform transmission lines.

A. Nth Order Binomial Form Nonuniform Transmission Line

In this case, the characteristic impedance distribution is given by

$$W_b(x)=W_0\left(1+\frac{1}{h}\cdot\frac{x}{l}\right)^n. \tag{16}$$

We obtain $k_b(x)$ of (6) and $Z_b(x)$ of (7) as follows:

$$k_b(x)=1+\frac{W_0}{L_0}\cdot\frac{h}{n+1}\left\{\left(1+\frac{1}{h}\cdot\frac{x}{l}\right)^{n+1}-1\right\} \tag{17}$$

$$Z_b(x)=\frac{W_0\left(1+\frac{1}{h}\cdot\frac{x}{l}\right)^n}{\left[1+\frac{W_0}{L_0}\cdot\frac{h}{n+1}\left\{\left(1+\frac{1}{h}\cdot\frac{x}{l}\right)^{n+1}-1\right\}\right]^2}. \tag{18}$$

Elements of a chain matrix $[F_b]$ of nth order binomial form nonuniform transmission line are given as follows [11]:

$$[F_b]=\begin{bmatrix}\mathbb{A}_b & \mathbb{B}_b\\ \mathbb{C}_b & \mathbb{D}_b\end{bmatrix}. \tag{19}$$

1) Case 1; n = odd:

$$\mathbb{A}_b=\frac{T}{M}\cdot\{J_{(n+1)/2}(h\beta l)\cdot N_{(n-1)/2}((1+h)\beta l) - N_{(n+1)/2}(h\beta l)\cdot J_{(n-1)/2}((1+h)\beta l)\} \tag{20}$$

$$\mathbb{B}_b=jW_0TM\cdot\{J_{(n+1)/2}(h\beta l)\cdot N_{(n+1)/2}((1+h)\beta l) - N_{(n+1)/2}(h\beta l)\cdot J_{(n+1)/2}((1+h)\beta l)\} \tag{21}$$

$$\mathbb{C}_b=j\frac{1}{W_0}\cdot\frac{T}{M}\cdot\{J_{(n-1)/2}(h\beta l)\cdot N_{(n-1)/2}((1+h)\beta l) - N_{(n-1)/2}(h\beta l)\cdot J_{(n-1)/2}((1+h)\beta l)\} \tag{22}$$

$$\mathbb{D}_b=TM\cdot\{N_{(n-1)/2}(h\beta l)\cdot J_{(n+1)/2}((1+h)\beta l) - J_{(n-1)/2}(h\beta l)\cdot N_{(n+1)/2}((1+h)\beta l)\} \tag{23}$$

where

$J_n(\beta l)$ Bessel function,
$N_n(\beta l)$ Neumann function

and

$$\left.\begin{aligned}T&=\sqrt{\frac{\pi h\beta l}{2}}\sqrt{\frac{\pi(1+h)\beta l}{2}}\\ M&=\left(\frac{1+h}{h}\right)^{n/2}\end{aligned}\right\}. \tag{24}$$

2) Case 2; n = even;

$$\mathbb{A}_b=\frac{T'}{M}\cdot\{J_{-(n+1)/2}(h\beta l)\cdot J_{(n-1)/2}((1+h)\beta l) + J_{(n+1)/2}(h\beta l)\cdot J_{-(n-1)/2}((1+h)\beta l)\} \tag{25}$$

$$\mathbb{B}_b=jW_0T'M\cdot\{J_{-(n+1)/2}(h\beta l)\cdot J_{(n+1)/2}((1+h)\beta l) - J_{(n+1)/2}(h\beta l)\cdot J_{-(n+1)/2}((1+h)\beta l)\} \tag{26}$$

$$\mathbb{C}_b=j\frac{1}{W_0}\cdot\frac{T'}{M}\cdot\{J_{(n-1)/2}(h\beta l)\cdot J_{-(n-1)/2}((1+h)\beta l) - J_{-(n-1)/2}(h\beta l)\cdot J_{(n-1)/2}((1+h)\beta l)\} \tag{27}$$

$$\mathbb{D}_b=T'M\cdot\{J_{-(n-1)/2}(h\beta l)\cdot J_{(n+1)/2}((1+h)\beta l) + J_{(n-1)/2}(h\beta l)\cdot J_{-(n+1)/2}((1+h)\beta l)\} \tag{28}$$

where

$$T'=(-1)^{n/2}\sqrt{\frac{\pi h\beta l}{2}}\sqrt{\frac{\pi(1+h)\beta l}{2}}. \tag{29}$$

Therefore, a chain matrix of the nonuniform transmission line with $Z_b(x)$ of (18) is the same expression as (13) with the W subscripts changed to b.

If the following relation is satisfied

$$\frac{W_0}{L_0}=\frac{n+1}{h} \tag{30}$$

then (18) simplifies as follows:

$$Z_b(x)=\frac{W_0}{\left(1+\frac{1}{h}\cdot\frac{x}{l}\right)^{n+2}}. \tag{31}$$

Namely, in this particular case, $(n+2)$th order convergent type binomial form transmission line can be obtained from nth order divergent type binomial form transmission line.

B. Exponential Transmission Line

The characteristic impedance distribution is given by

$$W_e(x)=W_0\exp(\delta x). \tag{32}$$

From (6) and (7) we get

$$k_e(x)=1+\frac{1}{\delta l}\cdot\frac{W_0}{L_0}\{\exp(\delta x)-1\} \tag{33}$$

and

$$Z_e(x)=\frac{W_0\exp(\delta x)}{\left[1+\frac{1}{\delta l}\cdot\frac{W_0}{L_0}\{\exp(\delta x)-1\}\right]^2}. \tag{34}$$

By using a chain matrix of the exponential transmission line, a chain matrix of the nonuniform transmission line with $Z_e(x)$ of (34) is given as the same expression as (13) with the W subscripts changed to e. Elements of a chain matrix $[F_e]$ of the exponential transmission line are given as follows [14]:

$$[F_e]=\begin{bmatrix}\mathbb{A}_e & \mathbb{B}_e\\ \mathbb{C}_e & \mathbb{D}_e\end{bmatrix} \tag{35}$$

$$\mathbb{A}_e=\frac{1}{N}\left(\cosh\Gamma l+\frac{\delta l}{2}\cdot\frac{\sinh\Gamma l}{\Gamma l}\right) \tag{36}$$

$$\mathbb{B}_e=jW_0N\cdot\beta l\cdot\frac{\sinh\Gamma l}{\Gamma l} \tag{37}$$

$$\mathbb{C}_e=j\frac{1}{W_0}\cdot\frac{1}{N}\cdot\beta l\cdot\frac{\sinh\Gamma l}{\Gamma l} \tag{38}$$

$$\mathbb{D}_e=N\left(\cosh\Gamma l-\frac{\delta l}{2}\cdot\frac{\sinh\Gamma l}{\Gamma l}\right) \tag{39}$$

where

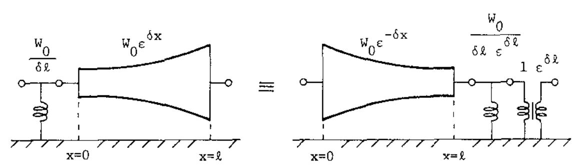
$$\left.\begin{aligned}\Gamma l&=\sqrt{\left(\frac{\delta l}{2}\right)^2-(\beta l)^2}\\ N&=\exp\left(\frac{\delta l}{2}\right)\end{aligned}\right\}. \tag{40}$$

If the following relation is applicable:

$$\frac{W_0}{L_0}=\delta l \tag{41}$$

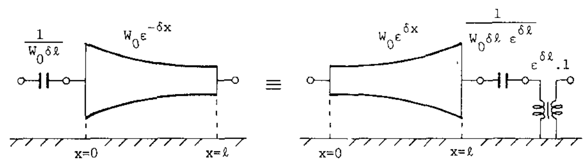
then $Z_e(x)$ of (34) becomes the characteristic impedance distribution of the convergent type exponential transmission line

$$Z_e(x)=W_0\exp(-\delta x). \tag{42}$$



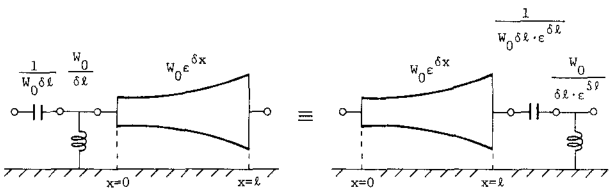
(a) (b)

Fig. 4. Extended Kuroda's identity for the exponential transmission line.



(a) (b)

Fig. 5. Extended Kuroda's identity for the exponential transmission line, the dual of circuits shown in Fig. 4.



(a) (b)

Fig. 6. Extended Kuroda's identity for lumped reactance elements and the exponential transmission line.

As shown in Fig. 4, in this particular case, the circuit consisting of the shunt lumped inductance and the divergent type exponential transmission line ($\delta>0$) is equal to the one consisting of a cascade connection of the convergent type exponential transmission line, the shunt lumped inductance and an ideal transformer. Similarly, we obtain Kuroda's identity shown in Fig. 5. Therefore, from Figs. 4 and 5, extended Kuroda's identity shown in Fig. 6 is also obtained. Namely, a lumped reactance circuit may be shifted through the exponential transmission line.

C. Hyperbolic Secant Squared Tapered Transmission Line

The characteristic impedance distribution $W_{sh}(x)$ of hyperbolic secant squared tapered transmission line is given by

$$W_{sh}(x)=W_0\,\mathrm{sech}^2(\delta x). \tag{43}$$

From (6) and (7) we obtain

$$k_{sh}(x)=1+\frac{1}{\delta l}\cdot\frac{W_0}{L_0}\tanh(\delta x) \tag{44}$$

and

$$Z_{sh}(x)=\frac{W_0\,\mathrm{sech}^2(\delta x)}{\left[1+\frac{1}{\delta l}\cdot\frac{W_0}{L_0}\tanh(\delta x)\right]^2}. \tag{45}$$

Elements of a chain matrix $[F_{sh}]$ of the hyperbolic secant squared tapered transmission line are given as follows [15]:

$$[F_{sh}]=\begin{bmatrix}\mathbb{A}_{sh} & \mathbb{B}_{sh}\\ \mathbb{C}_{sh} & \mathbb{D}_{sh}\end{bmatrix} \tag{46}$$

$$\mathbb{A}_{sh}=\cosh(\delta l)\cdot\cosh(\Gamma l)-\delta l\cdot\sinh(\delta l)\cdot\frac{\sinh(\Gamma l)}{\Gamma l} \tag{47}$$

$$\mathbb{B}_{sh}=jW_0\cdot\frac{\beta l}{\cosh(\delta l)}\cdot\frac{\sinh(\Gamma l)}{\Gamma l} \tag{48}$$

$$\mathbb{C}_{sh}=j\frac{1}{W_0}\cdot\frac{\Gamma l}{\beta l}\left\{\delta l\cdot\sinh(\delta l)\cdot\frac{\cosh(\Gamma l)}{\Gamma l}-\cosh(\delta l)\cdot\sinh(\Gamma l)\right\} \tag{49}$$

$$\mathbb{D}_{sh}=\frac{\cosh(\Gamma l)}{\cosh(\delta l)} \tag{50}$$

where

$$\Gamma l=\sqrt{(\delta l)^2-(\beta l)^2}. \tag{51}$$

A chain matrix of the nonuniform transmission line with $Z_{sh}(x)$ of (45) is given as the same expression as (13) with the W subscripts changed to *sh*.

If the following relation is satisfied

$$\frac{W_0}{L_0}=\delta l \tag{52}$$

then $Z_{sh}(x)$ of (45) becomes the characteristic impedance distribution of the convergent type exponential transmission line

$$Z_{sh}(x)=W_0\exp(-2\delta x). \tag{53}$$

D. Hermite Line

In this case, the characteristic impedance distribution is

$$W_H(x)=W_0\exp\left[(\delta x)^2\right]. \tag{54}$$

We obtain $k_H(x)$ of (6) and $Z_H(x)$ of (7) as follows:

$$k_H(x)=1+\frac{1}{\delta l}\cdot\frac{W_0}{L_0}\cdot\sum_{m=0}^{\infty}\frac{(\delta x)^{2m+1}}{(2m+1)\cdot m!} \tag{55}$$

and

$$Z_H(x)=\frac{W_0\exp\left[(\delta x)^2\right]}{\left[1+\frac{1}{\delta l}\cdot\frac{W_0}{L_0}\cdot\sum_{m=0}^{\infty}\frac{(\delta x)^{2m+1}}{(2m+1)\cdot m!}\right]^2}. \tag{56}$$

Therefore, by using the chain matrix of Hermite line [9], the chain matrix of the nonuniform transmission line with $Z_H(x)$ of (56) can be obtained as the same expression as (13).

By applying extended Kuroda's identity shown in Fig. 3 and repeating the same procedure for nonuniform transmission lines in examples III-A–III-D, we may obtain exact network functions of a number of nonuniform transmission lines.

IV. Conclusion

We have shown that Kuroda's identity can be extended to circuits consisting of lumped reactance elements and general nonuniform transmission lines, by considering the limit case of uniform transmission lines ($n\to\infty$). A characteristic impedance distribution $Z(x)$ of a transformed nonuniform transmission line may be uniquely obtained from a characteristic impedance distribution $W(x)$ of an original nonuniform transmission line. Also, if an exact network function of an original nonuniform transmission line is known, a network function of a transformed nonuniform transmission line can be exactly derived from the equivalent circuit. Finally, we applied extended Kuroda's identity to several circuits consisting of lumped inductances and nonuniform transmission lines, and get exact network functions of transformed nonuniform transmission lines.

Appendix
Derivation of (5)

Substituting (1) in (4):

$$\begin{aligned}k_j&=1+\frac{W_0}{L}\sum_{i=1}^{j}\left[1+\frac{a_1}{n}(i-1)+\frac{a_2}{n^2}(i-1)^2+\cdots+\frac{a_m}{n^m}(i-1)^m+\cdots\right]\\
&=1+\frac{W_0}{L}\left[\sum_{i=1}^{j}1+\frac{a_1}{n}\sum_{i=1}^{j}(i-1)+\frac{a_2}{n^2}\sum_{i=1}^{j}(i-1)^2+\cdots+\frac{a_m}{n^m}\sum_{i=1}^{j}(i-1)^m+\cdots\right]\\
&=1+\frac{W_0}{L}\left[j+\frac{a_1}{n}\left\{\frac{(j-1)^2}{2}+\frac{(j-1)}{2}\right\}+\frac{a_2}{n^2}\left\{\frac{(j-1)^3}{3}+\frac{(j-1)^2}{2}+\frac{(j-1)}{6}\right\}\right.\\
&\quad+\frac{a_3}{n^3}\left\{\frac{(j-1)^4}{4}+\frac{(j-1)^3}{2}+\frac{(j-1)^2}{4}\right\}+\cdots\\
&\quad\left.+\frac{a_m}{n^m}\left\{\frac{(j-1)^{m+1}}{m+1}+\frac{(j-1)^m}{2}+\cdots\right\}+\cdots\right].\end{aligned} \tag{5}$$

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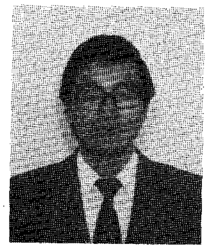

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